The former is more aesthetic, but the recent experimental results of the high-energy proton-proton and pion-proton scatterings[22] are favorable to the latter.

ACKNOWLEDGMENT

The author is very grateful to Dr. Y. Shimamoto for his kind interest.

APPENDIX

We shall derive (6.5)–(6.7) from (6.3). Because of the symmetry of (6.3) with respect to (l,n) and (k,r), it is sufficient to consider the case $l\geq k$ only. Carrying out the integration over z_0', we have

$$I=\theta(\kappa-z_0)\kappa^l z_0^{-l-1}\,\mathrm{Pf}\int_{z_0/\kappa}^{1} dy_0 \times y_0^{l-k-1}(\ln 1/y_0)^r(\ln\kappa y_0/z_0)^n. \quad \text{(A1)}$$

Putting

$$y_0=\exp(-x\ln\kappa/z_0)\,, \quad \text{(A2)}$$

we obtain

$$I=\theta(\kappa-z_0)\kappa^l z_0^{-l-1}(\ln\kappa/z_0)^{n+r+1}J\,, \quad \text{(A3)}$$

where

$$J\equiv\mathrm{Pf}\int_0^1 dx\, x^r(1-x)^n\exp[-x(l-k)\ln\kappa/z_0]. \quad \text{(A4)}$$

In the case $l=k$, (A3) with (A4) immediately leads to (6.6). For $l>k$, the exponential in (A4) becomes extremely small except for $x\simeq 0$ because we are interested in $z_0\simeq 0$. Hence we may approximate J by replacing $(1-x)^n$ by 1.[23] Then we obtain (6.5).

[22] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **10**, 376 (1963).

[23] The analytic continuation with respect to n and r does not invalidate this approximation. This can be checked by expanding $(1-x)^n$ into the Taylor series and integrating term by term.

PHYSICAL REVIEW VOLUME 133, NUMBER 5B 9 MARCH 1964

Analytic Continuation in Complex Angular Momentum and Integral Equations*

George Tiktopoulos

Palmer Physical Laboratory, Princeton University, Princeton, New Jersey†

(Received 14 October 1963)

An attack is made on the problem of the analytic continuation in the angular momentum variable l of amplitudes defined by integral equations beyond the value of Rel at which the kernel ceases to be of the Schmidt type and the Fredholm theory cannot be applied. A general technique is developed and applied to the Yukawa potential case and to the ladder graph series in the φ^3 theory. In both cases meromorphy is established for Re$l>-\frac{5}{2}$ and a procedure is indicated for a stepwise continuation to the entire l plane.

1. INTRODUCTION

THE importance of analyticity properties of scattering amplitudes in the complex angular momentum variable has motivated certain field theoretic approximations[1–13] in the framework of simple models. These models consist essentially of series of Feynman graphs studied either on the basis of integral equations[1–5] or by obtaining asymptotic forms[6–13] for large values of the momentum transfer.

The present situation indicates that sets of *planar*[1–11] Feynman graphs lead to [interpolating] partial-wave amplitudes $F(s,l)$ which are *meromorphic* in l at least in some region beyond the analyticity domain specified by the number of subtractions in the momentum transfer t. With the exception of the "superconvergent" φ^3 theory, one obtains in addition fixed branch points, e.g., in the φ^4 case and in the model of scalar particles interacting through the exchange of vector mesons.[3,5,9] These fixed branch points seem to be quite analogous to those appearing in the r^{-2} potential case.

This close analogy with the nonrelativistic potential case is more or less expected because for planar graphs the ρ_{tu} spectral function vanishes as in the case of

* Work supported in part by the Atomic Energy Commission and in part by the U. S. Air Force Office of Scientific Research, Air Research and Development Command.

† This work was begun during the author's stay at the Enrico Fermi Institute for Nuclear Studies and the Department of Physics, University of Chicago, Chicago, Illinois.

[1] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 and 2274 (1962).

[2] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962).

[3] R. F. Sawyer, Phys. Rev. **131**, 1384 (1963).

[4] N. Nakanishi, Phys. Rev. **130**, 1230 (1963).

[5] P. Suranyi, Phys. Letters **6**, 59 (1963).

[6] M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962).

[7] J. C. Polkinghorne, J. Math. Phys. **4**, 503 (1963).

[8] P. G. Federbush and M. T. Grisaru, Ann. Phys. (N.Y.) **22**, 263 and 299 (1963).

[9] J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963).

[10] I. G. Halliday, Nuovo Cimento **30**, 177 (1963).

[11] G. Tiktopoulos, Phys. Rev. **131**, 480 and 2373 (1963).

[12] J. C. Polkinghorne (to be published).

[13] S. Mandelstam (to be published).

amplitudes derived from superpositions of Yukawa potentials.

In contrast, from contributions of *nonplanar*[11–13] graphs ($\rho_{tu}\neq 0$), we expect and obtain new features specific for relativistic theories with crossing properties. Such are the Gribov-Pomeranchuk[14] singularities and moving (energy-dependent) branch points in the l plane.

The analytic continuation to the left-hand l plane for Yukawa potentials has been discussed by a number of authors by configuration space methods.[15–17] However, the momentum space integral equation approach, apart from manifest similarity to the field theoretic case, appears to have certain advantages as, for example, in connection with the study of the $|l|\to\infty$ behavior.[18] In Ref. 18 the Fredholm theory was used to establish the meromorphy in l in the region $\mathrm{Re}\,l>-\frac{3}{2}$ for Yukawa potentials.

In the more interesting case of the ladder graph series in φ^3 theory, Lee and Sawyer[1] have shown the meromorphy in the region $\mathrm{Re}\,l>-\frac{3}{2}$ also by Fredholm methods.

The purpose of this paper is to obtain some further information on the l-plane properties of amplitudes defined by integral equations. We develop a method for the analytic continuation of the solution beyond the value of $\mathrm{Re}\,l$ ($-\frac{3}{2}$ in the mentioned cases) at which the kernel ceases to be of the Schmidt type and the Fredholm theory cannot be directly applied.

The idea is to decompose the kernel of the integral equation

$$\psi=\varphi+K\psi$$

in two parts $K=S+R$, where S is of finite rank and R is of the Schmidt type in a larger domain of the l plane than K. We then use operator identities as, for example,

$$(1-K)^{-1}\varphi=(1-\Gamma R)^{-1}[1-R(1-\Gamma R)^{-1}]^{-1}\times R(1-\Gamma R)^{-1}(1+\Gamma)\varphi+(1-\Gamma R)^{-1}(1+\Gamma)\varphi$$
$$\Gamma=(1-S)^{-1}-1\,,$$

in which, after an *explicit* analytic continuation of the occurring operator products, the only resolvents involved are those of meromorphic Schmidt operators in the enlarged domain. The fact that such resolvents are *also* meromorphic under certain conditions is established by a theorem given in the Appendix. These conditions are essentially that the Laurent coefficients of the Schmidt operators should be of finite rank (separable). This turns out to be true for the Yukawa potential case and for the ladder series in φ^3 theory which are treated in Secs. II and III. The meromorphy of the partial-wave amplitude is established in both cases for $\mathrm{Re}\,l>-\frac{5}{2}$ and a general procedure is indicated for a further continuation to the entire l plane.

[14] V. N. Gribov and I. Pomeranchuk, Phys. Letters 2, 239 (1962).

[15] S. Mandelstam, Ann. Phys. (N.Y.) 19, 254 (1959).

[16] H. A. Bethe and T. Kinoshita, Phys. Rev. 128, 1418 (1962).

[17] M. Froissart, J. Math. Phys. 3, 922 (1962).

[18] L. Brown, D. I. Fivel, B. W. Lee, and R. F. Sawyer (to be published).

2. THE YUKAWA CASE

The partial-wave Lippman-Schwinger equation for the potential $gr^{-1}e^{-\mu r}$ is

$$\psi_l(p,s)=\frac{-g}{2s^{1/2}}Q_l\left(\frac{p^2+s+\mu^2}{2ps^{1/2}}\right)+\frac{g}{\pi}\int_0^\infty Q_l\left(\frac{p^2+p'^2+\mu^2}{2pp'}\right)\psi_l(p',s)\frac{dp'}{p'^2-s}$$

where $\psi_l(p,s)$ is the off-the-energy-shell, partial-wave amplitude and s is the squared c.m. momentum. The physical amplitude is given by $\psi_l[s^{1/2},s]$. If s is not nonnegative real[19] and $\mathrm{Re}\,l>-\frac{3}{2}$, the inhomogeneous term is square integrable and the kernel is of the Schmidt type (for convenience we take $d\mu(p)=(p^2-s)^{-1}dp$ as a measure of integration). The kernel is also analytic in $\mathrm{Re}\,l>-\frac{3}{2}$ except for a simple pole at $l=-1$ whose residue is of finite rank. By means of the Fredholm theory,[18] or simply according to the general theorem given in the Appendix, we conclude that the resolvent and therefore $\psi_l(p,s)$ is meromorphic in the region $\mathrm{Re}\,l>-\frac{3}{2}$.

Let us now apply our method to enlarge this meromorphy domain. Our equation is symbolically

$$\psi=\varphi+K\psi.$$

We decompose our kernel as follows

$$K(p,p')=\frac{g}{\pi}Q_l\left(\frac{p^2+p'^2+\mu^2}{2pp'}\right)=S(p,p')+R(p,p')\,,\quad (1)$$

where

$$S(p,p')=c(l)\left[\left(\frac{p^2+\mu^2}{pp'}\right)^{-l-1}+\left(\frac{p'^2+\mu^2}{pp'}\right)^{-l-1}-\left(\frac{\mu^2}{pp'}\right)^{-l-1}\right],\quad (2)$$

$$c(l)=\frac{g}{\pi^{1/2}}\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}$$

is of finite rank.

Using the operator identities

$$(1-X-Y)^{-1}=[1-(1-X)^{-1}Y]^{-1}(1-X)^{-1},\quad (\mathrm{A})$$

$$(1-XY)^{-1}=1+X(1-YX)^{-1}Y\,,\quad (\mathrm{B})$$

we obtain

$$(1-K)^{-1}=(1-\Gamma R)^{-1}[1-R(1-\Gamma R)^{-1}]^{-1}\times R(1-\Gamma R)^{-1}(1+\Gamma)+(1-\Gamma R)^{-1}(1+\Gamma)\,,\quad (3)$$

where

$$\Gamma=(1-S)^{-1}-1\,.$$

[19] We shall not be concerned with analyticity properties in the energy variable in this paper.

In terms of the two vectors

$$u_1=[(p^2+\mu^2)/p]^{-l-1},\quad u_2=(1/p)^{-l-1},$$

we have[20]

$$S=c(l)|u_1\rangle\langle u_2^*|+c(l)|u_2\rangle\langle u_1^*|+c(l)(\mu^2)^{-l-1}|u_2\rangle\langle u_2^*|$$
$$=c(l)|u_1\rangle\langle u_2^*|+c(l)|u_2\rangle\langle u_1^*+(\mu^2)^{-l^*-1}u_2^*|\,.$$

The resolvent $1+\Gamma=(1-S)^{-1}$ can be computed explicitly in the form

$$\Gamma=|u_1\rangle\langle a_1^*(l)u_1^*+a_2^*(l)u_2^*|+|u_2\rangle\langle b_1^*(l)u_1^*+b_2^*(l)u_2^*|$$

where the coefficients a_1, a_2, b_1, and b_2 are simple rational functions of the inner products $\langle u_i^*|u_j\rangle$.

For example:

$$a_1(l)=c^2(l)\langle u_2^*|u_2\rangle\begin{vmatrix} c\langle u_2^*|u_1\rangle-1 & c\langle u_2^*|u_2\rangle \\ c\langle u_1^*+(\mu^2)^{-l^*-1}u_2^*|u_1\rangle & c\langle u_1^*+(\mu^2)^{-l^*-1}u_2^*|u_2\rangle-1\end{vmatrix}^{-1}.$$

The vectors u_1, u_2 are square integrable for $\mathrm{Re}l>-\frac{3}{2}$ only. However, the inner products $\langle u_i|u_j\rangle$ can be analytically continued to the left of $\mathrm{Re}l=-\frac{3}{2}$. We have[21]

$$\langle u_1^*|u_2\rangle=\int_0^\infty(p^2+\mu^2)^{-l-1}(p^2-s)^{-1}p^{2l+2}dp=\tfrac{1}{2}(\mu^2)^{-l-1}(-s)^{l+1/2}\frac{\Gamma(l+\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(l+2)}F\left(l+1;l+\tfrac{3}{2};l+2;1+\frac{s}{\mu^2}\right),$$

$$\langle u_2^*|u_2\rangle=\int_0^\infty(p^2-s)^{-1}p^{2l+2}dp=\tfrac{1}{2}(-s)^{l+1/2}\Gamma(l+\tfrac{3}{2})\Gamma(-l-\tfrac{1}{2}),$$

$$\langle u_1^*|u_1\rangle=\int_0^\infty(p^2+\mu^2)^{-2l-2}(p^2-s)^{-1}p^{2l+2}dp=\tfrac{1}{2}(\mu^2)^{-2l-2}(-s)^{l+1/2}\frac{[\Gamma(l+\frac{3}{2})]^2}{\Gamma(2l+3)}F\left(2l+2;l+\tfrac{3}{2};2l+3;1+\frac{s}{\mu^2}\right).$$

Thus the coefficients a_1, a_2, b_1, b_2 are meromorphic in the entire l plane. In the strip $-\frac{5}{2}<\mathrm{Re}l<-\frac{3}{2}$ for $g\to 0$ we have a pole approaching $l=-2$. We note that the simple poles of the hypergeometric functions at $l=-\frac{3}{2}$ are cancelled by the corresponding zeroes of $c(l)$.

Let us now consider the finite rank operator ΓR

$$\Gamma R=|u_1\rangle\langle a_1^*(l)u_1^*+a_2^*(l)u_2^*|R+|u_2\rangle\langle b_1^*(l)u_1^*+b_2^*(l)u_2^*|R.$$

We have to continue analytically the vectors $\langle u_1|R$ and $\langle u_2|R$. Using the expansion

$$Q_l(z)=\pi^{1/2}\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}(2z)^{-l-1}+\pi^{1/2}\frac{\Gamma(l+3)}{\Gamma(l+\frac{5}{2})}(2z)^{-l-3}+\cdots,$$

we obtain

$$R(p,p')=c(l)\left[\left(\frac{p^2+p'^2+\mu^2}{pp'}\right)^{-l-1}-\left(\frac{p^2+\mu^2}{pp'}\right)^{-l-1}-\left(\frac{p'^2+\mu^2}{pp'}\right)^{-l-1}+\left(\frac{\mu^2}{pp'}\right)^{-l-1}\right]$$
$$+\frac{g}{\pi^{1/2}}\frac{\Gamma(l+3)}{\Gamma(l+\frac{5}{2})}\left(\frac{p^2+p'^2+\mu^2}{pp'}\right)^{-l-3}+\cdots.\quad(4)$$

Only the expression in brackets may produce singularities of $\langle u_i|R$ in the strip $-\frac{5}{2}<\mathrm{Re}l<-\frac{3}{2}$. The first term in $\langle u_1|R$ is

$$c(l)p'^{l+1}\int_0^\infty(q^2+\mu^2)^{-l-1}(q^2+p'^2+\mu^2)^{-l-1}(q^2-s)^{-1}q^{2l+2}dq.$$

Since

$$(q^2-s)^{-1}=(q^2+\mu^2)^{-1}+(\mu^2+s)(q^2-s)^{-1}(q^2+\mu^2)^{-1},$$

and we are interested in the region $q\to\infty$ (the contribution from $q\to 0$ is canceled by the contribution of the third

[20] The star denotes complex conjugation.

[21] *Bateman Manuscript Project, Tables of Integral Transforms,* edited by H. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 1, p. 310.

term in the brackets), we consider[21]

$$c(l)p'^{l+1}\int_0^\infty (q^2+\mu^2)^{-l-2}(q^2+p'^2+\mu^2)^{-l-1}q^{2l+2}dq=\frac{c(l)}{2}(p'^2+\mu^2)^{1/2}p'^{-l-3}\frac{\Gamma(l+\frac{3}{2})\Gamma(-\frac{1}{2})}{\Gamma(l+1)}F\left(l+2,-l;\tfrac{3}{2};-\frac{\mu^2}{p'^2}\right)$$
$$+\frac{c(l)}{2}(\mu^2)^{-1/2}(p'^2+\mu^2)^{1/2}p'^{-l-2}\frac{\Gamma(l+\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(l+2)}F\left(l+\tfrac{3}{2},-l-\tfrac{1}{2};\tfrac{1}{2};-\frac{\mu^2}{p'^2}\right). \quad (5)$$

Clearly, the leading contribution for $p'\to\infty$ is canceled by that of the third term in (4) so that the continued vector $\langle u_1|R$ behaves like p'^{-l-2} as $p'\to\infty$. Also from Eq. (4) it is seen that $\langle u_1|R$ behaves like p'^{l+3} as $p'\to 0$. Therefore the continued vector $\langle u_1|R$ is meromorphic in l and is square-integrable in the strip $-\frac{5}{2}<\mathrm{Re}l<-\frac{3}{2}$. It has only a simple pole at $l=-2$ since the poles of the hypergeometric functions at $l=-\frac{3}{2}$ are again canceled by the zero of $c(l)$.

The operator $(1-\Gamma R)^{-1}-1$ is a linear combination of the projections $|u_i\rangle\langle u_j^*|R$ whose coefficients are simple rational functions of the inner products $\langle u_j^*|R|u_i\rangle$. These latter are easily seen to be meromorphic in the strip $-\frac{5}{2}<\mathrm{Re}l<-\frac{3}{2}$ since the only divergence comes from the leading contribution of the first term on right-hand side of Eq. (5), which produces an additional pole at $l=-2$.

Setting

$$\Gamma=\sum_{i,j=1,2} b_{ij}|u_i\rangle\langle u_j^*|$$

$$[1-\Gamma R]^{-1}=1+\sum_{i,j=1,2} a_{ij}|u_i\rangle\langle u_j^*|R,$$

we use the identity (3) to obtain

$$\begin{aligned}[1-K]^{-1}|\varphi\rangle &=[1+\textstyle\sum a_{ij}|u_i\rangle\langle u_j^*|R][1-R-\sum a_{i,j}R|u_i\rangle\langle u_j^*|R]^{-1}\\ &\quad\times[R+\textstyle\sum a_{ij}R|u_i\rangle\langle u_j|R][1+\sum b_{ij}|u_i\rangle\langle u_j^*|]|\varphi\rangle\\ &\quad+[1+\textstyle\sum a_{ij}|u_i\rangle\langle u_j^*|][1+\sum b_{ij}|u_i\rangle\langle u_j^*|]|\varphi\rangle.\end{aligned}$$

Since $R+\sum a_{ij}R|u_i\rangle\langle u_j^*|R$ is of the Schmidt type in the region $\mathrm{Re}l>-\frac{5}{2}$ and satisfies the conditions of the theorem given in the Appendix, its resolvent is meromorphic in that region. Furthermore the residues of the resolvent are of finite rank. Since the inner products $\langle u_i^*|u_j\rangle$ have been shown to be meromorphic already, the only analytic continuation needed in our expression for $[1-K]^{-1}\varphi$ is that of $\langle u_i^*|\varphi\rangle$. This can be done quite similarly to $\langle u_i^*|u_j\rangle$.

We have thus established the meromorphy of the partial-wave amplitude for $\mathrm{Re}l>-\frac{5}{2}$. The poles are given by the zeros of the Fredholm denominator

$$\det[1-R-\textstyle\sum a_{ij}R|u_i\rangle\langle u_j^*|R]=0.$$

Expansions in powers of the coupling constant can be obtained from this equation for the Regge trajectories.

Our method may be used for a stepwise analytic continuation to the entire left-hand l plane, although taking more terms of the Q_l expansion becomes rather tedious.

Let us now consider the case of a superposition of Yukawa potentials

$$gr^{-1}\int_{\mu_0}^\infty \sigma(\mu)e^{-\mu r}d\mu.$$

Clearly, the applicability of our method depends on the behavior of $\sigma(\mu)$ as $\mu\to\infty$. Let us take

$$\sigma(\mu)\underset{\mu\to\infty}{\sim}\mu^\beta.$$

The leading term in our kernel will be proportional to

$$(pp')^{l+1}\int_{\mu_0}^\infty (p^2+p'^2+\mu^2)^{-l-1}\mu^\beta d\mu$$
$$=(pp')^{l+1}\frac{\mu_0^{-2l-1+\beta}}{2l-\beta+1}F\left(l+1,l-\frac{\beta-1}{2};\; l-\frac{\beta-3}{2};-\frac{p^2+p'^2}{\mu_0^2}\right).$$

For large values of $p^2+p'^2$ we have leading terms proportional to

$$(p^2+p'^2+\mu_0^2/pp')^{-l-1}\quad\text{and}\quad (p^2+p'^2+\mu_0^2/pp')^{-l+(\beta-1)/2}(pp')^{(\beta+1)/2}.$$

Since our integration measure is $(p^2-s)^{-1}dp$, we can easily verify that in a decomposition $K=S+R$ of the type given by Eqs. (1) and (2) the kernel R behaves better than K only if $\beta<0$.[22] *Thus the limit of applicability of our method is $\beta=0$ corresponding to an r^{-2} singularity of the potential at the origin.* It is well known that such potentials produce *branch cuts* in the l plane and can be treated trivially by incorporating the gr^{-2} term in the centrifugal term of the radial Schrödinger equation.

For $\beta=0$ the kernel is not of the Schmidt type for *any* value of l. In this and the analogous relativistic cases (see Introduction) the appearance of a (square-root) branch point in the l plane marks the breakdown of the Fredholm theory as expected, since the uniqueness of the Fredholm solution is incompatible with the two-valuedness due to the branch cut.

[22] The details of our calculations in this section remain unchanged if $\beta\leqslant -4$. For $-4<\beta<0$ these are certain straightforward modifications to be made. For example, the $\mu\to\infty$ behavior of $\sigma(\mu)$ produces a pole at $l=(\beta-1)/2$. As a result we have a Regge pole near $l=(\beta-1)/2$ in the weak coupling limit. (We have assumed here that $\int_{\mu_0}^\infty\sigma(\mu)\mu^\alpha d\mu$ is meromorphic.)

3. LADDER APPROXIMATION

Let us now turn to a simple field-theoretic model. We consider the amplitude given by the series of ladder graphs in the $g\varphi^3$ theory. After a counter-clockwise rotation of the integration contours[23] of the energy variables to the imaginary axis, we have the following partial-wave integral equation[1]:

$$T_l(p,\omega,s)=\frac{g^2}{(2\pi)^3}Q_l\left(\frac{p^2+p_s^2+(\omega-i\omega_s)^2+\mu^2}{2pp_s}\right)+\frac{g^2}{(2\pi)^3}\int_0^\infty\int_{-\infty}^\infty Q_l\left(\frac{p^2+p'^2+(\omega-\omega')^2+\mu^2}{2pp'}\right)\times T_l(p',\omega',s)\frac{dp'd\omega'}{\sigma(p',\omega')},$$

where

$$\sigma(p,\omega)=\left[p^2+\left(\omega-i\frac{s^{1/2}}{2}\right)^2+\mu^2\right]\left[p^2+\left(\omega+i\frac{s^{1/2}}{2}\right)^2+\mu^2\right]$$

and $s=4p_s^2+4\mu^2$ is the squared c.m. energy.

The meromorphy of $T_l(p,\omega,s)$ in the region[1] $\mathrm{Re}\,l>-\frac{3}{2}$ follows immediately by observing that the kernel is of the Schmidt type in that region and that it is analytic except for a simple pole at $l=-1$ whose residue is of finite rank.

Let us now apply our method in order to enlarge the meromorphy domain. The situation is somewhat complicated by the presence of the ω variables and some additional algebra is needed.

It is convenient to first make a similarity transformation

$$\psi(p,\omega)=p^{-l-1}T_l(p,\omega),$$

$$K(p,\omega;p',\omega')=\frac{g^2}{(2\pi)^3}\left(\frac{p}{p'}\right)^{-l-1}Q_l\left(\frac{p^2+p'^2+(\omega-\omega')^2+\mu^2}{2pp'}\right),$$

$$\varphi(p,\omega)=\frac{g^2}{(2\pi)^3}p^{-l-1}Q_l\left(\frac{p^2+p_s^2+(\omega-i\omega_s)^2+\mu^2}{2pp_s}\right).$$

Taking $d\tau(p,\omega)=dpd\omega/\sigma(p,\omega)$ as the measure of integration for inner products, we have the equation

$$\psi=\varphi+K\psi.$$

We now decompose K as follows

$$K=BW+R,$$

where

$$B(p,\omega;p',\omega')=c(l)(p^2+(\omega-\omega')^2+\mu^2)^{-l-1}\tau(\omega');\quad \tau(\omega)=\left[\int_0^\infty\frac{dg}{\sigma(g,\omega)}\right]^{-1},$$

$$W(p,\omega;p',\omega')=\delta(\omega-\omega')p'^{2l+2},$$

$$R(p,\omega;p',\omega')=c(l)[(p^2+p'^2+(\omega-\omega')^2+\mu^2)^{-l-1}-(p^2+(\omega-\omega')^2+\mu^2)^{-l-1}]p'^{2l+2}+\frac{g^2}{(2\pi)^3}\pi^{1/2}\frac{\Gamma(l+3)}{\Gamma(l+\frac{5}{2})}(p^2+p'^2+(\omega-\omega')^2+\mu^2)^{-l-3}p^2p'^{2l+4}+\cdots,$$

where

$$c(l)=\frac{g^2}{(2\pi)^3}\pi^{1/2}\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}.$$

We have

$$(1-K)^{-1}=[1-(1-BW)^{-1}R]^{-1}(1-BW)^{-1}.\tag{6}$$

Let us now decompose B in the following way:

$$B=S+M,$$

where

$$S(p,\omega;p',\omega')=c(l)(\omega'^2+\mu^2)^{-l-1}\tau(\omega'),$$

$$M(p,\omega;p',\omega')=c(l)[(p^2+(\omega-\omega')^2+\mu^2)^{-l-1}-(\omega'^2+\mu^2)^{-l-1}]\tau(\omega').$$

Using identities (A) and (B) we have

$$(1-BW)^{-1}=(1-SW)^{-1}+(1-SW)^{-1}M[1-W(1-SW)^{-1}M]^{-1}W(1-SW)^{-1},\tag{7}$$

[23] The contour rotation is strictly possible only for $|\mathrm{Re}\,s^{1/2}|<2\mu$. This difficulty has been pointed out by N. Nakanishi (Ref. 4). We shall not consider the problem of enlarging the s-plane domain in this paper.

where the following operator products occur (their kernels are indicated with the variable grouping $p,\omega;p',\omega'$):

$$SW\to c(l)(\omega'^2+\mu^2)^{-l-1}p'^{2l+2}$$

$$\Gamma=[1-SW]^{-1}-1\to a(l)(\omega'^2+\mu^2)^{-l-1}p'^{2l+2}$$

$$a(l)=\left[1-c(l)\int(\omega^2+\mu^2)^{-l-1}p^{2l+2}\frac{dpd\omega}{\sigma(p,\omega)}\right]^{-1}c(l)$$

$$W[1-SW]^{-1}\to p'^{2l+2}\delta(\omega-\omega')+a(l)\int\frac{q^{2l+2}dq}{\sigma(q,\omega)}(\omega'^2+\mu^2)^{-l-1}p'^{2l+2}$$

$$W[1-SW]^{-1}M\to c(l)\int[(q^2+(\omega-\omega')^2+\mu^2)^{-l-1}-(\omega'^2+\mu^2)^{-l-1}]\frac{q^{2l+2}dq}{\sigma(q,\omega)}\tau(\omega')$$
$$+c(l)a(l)\int\frac{\bar{q}^{2l+2}d\bar{q}}{\sigma(\bar{q},\omega)}\int(\nu^2+\mu^2)^{-l-1}[(q^2+(\nu-\omega')^2+\mu^2)^{-l-1}-(\omega'^2+\mu^2)^{-l-1}]\frac{q^{2l+2}dqd\nu}{\sigma(q,\nu)}\tau(\omega').$$

These operators can be shown by standard methods to be meromorphic in $\mathrm{Re}l>-\frac{5}{2}$. The analytic continuation of the integrals produces a pole at $l=-\frac{3}{2}$ because of the $q\to0$ behavior of the integrands. The corresponding residue is not of finite rank and thus the meromorphy of the resolvent would break down. However these poles are cancelled by the corresponding zeros of $c(l)$. The continuation of the integrals also produces a pole at $l=-2$ whose residue is of finite rank.

We now note that the continued operator

$$W[1-SW]^{-1}M$$

(after a trivial similarity transformation) is a Schmidt operator for $\mathrm{Re}l>-\frac{5}{2}$. This follows by writing

$$\int_0^\infty\frac{q^{2l+2}dq}{\sigma(q,\omega)}=(\omega^2)^{l-1}\int_0^\infty\frac{\omega^4}{\sigma(\omega q,\omega)}q^{2l+2}dq\underset{\omega\to\infty}{\longrightarrow}(\omega^2)^{l-1},\quad(8)$$

$$\int_0^\infty(q^2+(\omega-\omega')^2+\mu^2)^{-l-1}\frac{q^{2l+2}dq}{\sigma(q,\omega)}$$
$$=(\omega^2)^{-3/2}\int_0^\infty\left(q^2+\frac{(\omega-\omega')^2+\mu^2}{\omega^2}\right)^{-l-1}\frac{\omega^4}{\sigma(\omega q,\omega)}$$
$$\times q^{2l+2}dq\underset{\omega\to\infty}{\longrightarrow}(\omega^2)^{-3/2},\quad(9)$$

so that the kernel of $W[1-SW]^{-1}M$ is of the form

$$(\omega^2+\mu^2)^{-3/2}b(\omega,\omega')(\omega'^2+\mu^2)^{-l-3/2}\tau(\omega'),$$

where $b(\omega,\omega')$ is a bounded function. This is a Schmidt kernel for $\mathrm{Re}l>-\frac{5}{2}$ because the effective measure of integration is

$$\int_0^\infty\frac{dpd\omega}{\sigma(p,\omega)}=\frac{d\omega}{\tau(\omega)}.$$

Therefore, the resolvent $N=[1-W(1-SW)^{-1}M]^{-1}$ exists and is meromorphic in $\mathrm{Re}l>-\frac{5}{2}$ according to the Appendix theorem. Also the similarity transformation which makes $W(1-SW)^{-1}M$ a Schmidt operator shows that $(1-SW)^{-1}MN$ exists:

$$(1-SW)^{-1}MX[1-X^{-1}W(1-SW)^{-1}MX]^{-1}X^{-1}=(1-SW)^{-1}MN$$

with

$$X\to(\omega^2+\mu^2)^{l/2}\tau^{1/2}(\omega)\delta(\omega-\omega')$$
$$X^{-1}\to(\omega^2+\mu^2)^{-l/2}\tau^{3/2}(\omega)\delta(\omega-\omega').\quad(10)$$

Substituting (7) into (6) and using identities (A) and (B) we have:

$$(1-K)^{-1}\varphi$$
$$=\{1-(1+\Gamma)R-(1+\Gamma)MNW(1+\Gamma)R\}^{-1}\times(1-BW)^{-1}\varphi\quad(11)$$
$$=\{1+(1-A)^{-1}[1-(1+\Gamma)R(1-A)^{-1}]^{-1}(1+\Gamma)R\}\times(1-A)^{-1}(1-BW)^{-1}\varphi,$$

where

$$A=(1+\Gamma)MNW(1+\Gamma)R$$

so that

$$(1-A)^{-1}=1+(1+\Gamma)MN\times\{1-W(1+\Gamma)R(1+\Gamma)MN\}^{-1}W(1+\Gamma)R.\quad(12)$$

Expression (11) together with (12) is our final form for the solution. The purpose of the algebra is to bring the badly behaved operators between well-behaved ones (in particular R) so that analytic continuations of operator products can be explicitly carried out, resulting in Schmidt operators whose resolvents are meromorphic according to the Appendix theorem provided their Laurent coefficients are of finite rank.

Several such continuations are required in our expression for $(1-K)^{-1}\varphi$ given by (11) and (12). They can be carried out on the basis of integrals of type (8) and (9). For example

$$RM\to c^2(l)\int\{[q^2+p^2+(\omega-\nu)^2+\mu^2]^{-l-1}-[p^2+(\omega-\nu)^2+\mu^2]^{-l-1}\}$$
$$\times\{[q^2+(\nu-\omega')^2+\mu^2]^{-l-1}-(\omega'^2+\mu^2)^{-l-1}\}q^{2l+2}\frac{dqd\nu}{\sigma(q,\nu)}\tau(\omega').$$

It is easily verified that (i) the pole at $l=-\frac{3}{2}$ is cancelled by the zero of $c(l)$, (ii) the residue of the pole at $l=-2$ is of finite rank, and (iii) the behavior for large values of p, $|\omega|$ and $|\omega'|$ is at most p^{-2l-3}, $|\omega|^{-2l-3}$ and $|\omega'|^{-2l-3}$, respectively.

Also a (multiplicative) similarity transformation is needed in order to bring $(1+\Gamma)R(1-A)^{-1}$ to a proper Schmidt form

$$[1-(1+\Gamma)R(1-A)^{-1}]^{-1} = Y[1-Y^{-1}(1+\Gamma)R(1-A)^{-1}Y]^{-1}Y^{-1}$$

with

$$Y \to p^{-l-2}\sigma(p,\omega)\delta(p-p')\delta(\omega-\omega').$$

Furthermore simple power counting shows the existence of

$$(1-A)^{-1}[1-(1+\Gamma)R(1-A)^{-1}]^{-1}(1+\Gamma)R = (1-A)^{-1}Y\cdot[1-Y^{-1}(1+\Gamma)R(1-A)^{-1}Y]^{-1} \cdot Y^{-1}(1+\Gamma)R.$$

We have thus completed the proof of the meromorphy of the partial-wave amplitude for $\text{Re}l>-\frac{5}{2}$. Also the meromorphy in the coupling constant is readily established again by means of the Appendix theorem. Perturbation expansions for the Regge trajectories may be obtained from the equation

$$\det[1-(1+\Gamma)R(1-A)^{-1}]=0,$$

i.e., the vanishing of the Fredholm denominator for the kernel $(1+\Gamma)R(1-A)^{-1}$, although the convergence of such perturbation expansions is not guaranteed.

In principle, we see no essential difficulty in using the same procedure for a stepwise analytic continuation to the entire l plane. The main point is that after an appropriate algebraic manipulation of $(1-K)^{-1}$ one carries out explicit analytic continuations of operator-products. There arise poles at half-integer values of l from contributions of the small values of the momentum integration variables. These are canceled by the zeros of the coefficients in the expansion of Q_l. The poles arising from contributions of large values of the momentum and energy variables occur at the negative integer values of l and their residues are of finite rank due to the algebraic structure of the argument of Q_l.

In the case of ladder diagrams for the scattering of scalar particles exchanging vector mesons or bubble parts in a φ^4 theory,[3,5] the situation is similar to that of the r^{-2} potential case discussed at the end of Sec. II. The associated kernels are not of the Schmidt type for *any* value of l, so that the application of the method presented here must be preceded by a transformation of the type used by Suranyi[5] (Mellin transformation) to convert them to the Schmidt type.

ACKNOWLEDGMENT

I would like to thank Dr. P. Deliyannis of the Illinois Institute of Technology for helpful discussions.

APPENDIX

For the convenience of the reader we give the definitions of certain mathematical objects which occur in this paper.

We shall be working in a Hilbert space $\mathcal{H}$ of L_2 functions, i.e., complex square-integrable functions over a certain measure μ.

The *norm* of a vector φ in $\mathcal{H}$ is defined by

$$\|\varphi\|=[\langle\varphi|\varphi\rangle]^{1/2}=\left[\int|\varphi(x)|^2d\mu(x)\right]^{1/2}.$$

An operator A is *bounded* if there exists a (positive) number M such that

$$\|A\varphi\|<M\|\varphi\|$$

for *any* φ in H. The *bound* of a bounded operator is defined by

$$\|A\|=\sup\|A\varphi\|;\quad \|\varphi\|=1.$$

An operator A is *completely continuous* if given $\epsilon>0$, it can be decomposed as $A=S+N$ where S is of finite rank and $\|N\|<\epsilon$. A completely continuous operator is always bounded.

A *Schmidt operator* may be defined as an *integral* operator whose kernel satisfies the condition

$$\int\int|K(x,y)|^2d\mu(x)d\mu(y)<+\infty.$$

In books on integral equations[24] it is shown that Schmidt operators are completely continuous (the converse is not always true). Also the Fredholm series for the resolvent $(1-K)^{-1}$ of a Schmidt operator is valid.

Let us now define analyticity[25] for vectors and operators in our Hilbert space.

A family of vectors $\varphi(l)$ [operators $A(l)$] depending on a complex parameter l is *analytic* in some domain D of the l plane if $\langle\psi|\varphi(l)\rangle$ $[\psi_1|A(l)\psi_2\rangle]$ is an analytic function of l in D for every ψ $[\psi_1,\psi_2]$ in $\mathcal{H}$.

A sufficient condition for analyticity, which may be used for the operators occurring in this paper, is uniform (with respect to l) boundedness and analyticity of the kernel $K(x,y,l)$ in l for all x, y.

We can also define meromorphic vectors in D by the form

$$\varphi(l)=\varphi_0(l)+\sum_{i,n=1}^{N,M}\frac{1}{(l-l_i)^n}\varphi_{i,n},$$

where $\varphi_0(l)$ is analytic in D and $\varphi_{i,n}$ are l-independent vectors. Meromorphic operators are defined similarly.

---

[24] See for example R. Riesz and B. Nagy, *Functional Analysis* (Frederick Ungar Publishing Company, New York, 1955).

[25] See H. G. Garnir, *Les Problèmes aux limites de la physique mathematique* (Birkhäuser, Basel, 1958).

The following theorem[26] is more general than we need it in this paper. However this generality does not complicate the proof.

Theorem. Let $M(l)$ be a family of completely continuous operators of the form $M(l)=P(l)+A(l)$ where $P(l)$ is of finite rank and meromorphic in some connected domain D of the complex l plane and $A(l)$ is analytic in D. If $[1-M(l)]^{-1}$ exists at some point l_0 of D, then

(1) *$[1-M(l)]^{-1}$ exists and is meromorphic in D,*

(2) *$[1-M(l)]^{-1}=G(l)+B(l)$ where $G(l)$ is of finite rank[27] and $B(l)$ is analytic in D.*

Although all analytic finite-rank operators occurring in this paper are of the explicit form

$$S(l)=\sum_{i=1}^{N}|\psi_i(l)\rangle\langle\varphi_i(l)|\,, \tag{13}$$

where $\psi_i(l)$ and $\varphi_i^*(l)$ are analytic vectors (and N is l independent), we would like to show that this is always the case.

Lemma. Every operator family $S(l)$ which is analytic and of finite rank in a connected domain D of the l plane can be represented in the form (13).

Proof of the lemma. Let $u_1, u_2, \cdots$ be a basis in our Hilbert space $\mathcal{H}$. From the sequence $\{v_i(l)=S(l)u_i\}$ we select a subsequence $v_{i_1}, v_{i_2}, \cdots$ as follows: If the determinants

$$\begin{bmatrix} \langle u_{j_1}|v_{i_1}(l)\rangle & \langle u_{j_2}|v_{i_1}(l)\rangle\cdots & \langle u_{j_n}|v_{i_1}(l)\rangle \\ \langle u_{j_1}|v_{i_2}(l)\rangle & \cdots & \cdots \\ \cdots & & \\ \langle u_{j_1}|v_{i_{n-1}}(l)\rangle & & \\ \langle u_{j_1}|v_i(l)\rangle & \cdots & \langle u_{j_n}|v_i(l)\rangle \end{bmatrix}$$

vanish identically in D for $i<i_n$ and *all* $j_1, j_2, \cdots j_n$ and they do not for $i=i_n$, then $v_{i_n}(l)$ is the nth term of the subsequence. The analyticity of the determinants implies that the vectors of the subsequence are linearly independent in D with the exception of at most a countably infinite set of points. However $S(l)$ is of finite rank at *every* point of D. We conclude that our subsequence is finite and thus a finite set of analytic vectors $v_i(l)=S(l)u_i$ spans $S(l)\mathcal{H}$ for l in D.

An entirely analogous argument shows that a finite set of antianalytic vectors $w_i(l)=S^*(l)u_i$ spans $S^*(l)\mathcal{H}$. Thus we can write

$$S(l)=\sum_{i=1}^{N} c_i(l)\,|v_i(l)\rangle\langle w_i(l)|\,.$$

[26] We have not been able to find the equivalent of this theorem in the mathematical literature. A somewhat less general version was given (without the proof) by C. Lovelace at the 1963 Scottish Universities' Summer School for Physics, Edinburgh.

[27] In the complex angular momentum context of this paper, this assertion establishes the finite degeneracy of the Regge pole states.

The desired form is readily obtained since the analyticity of the coefficients $c_i(l)$ can be inferred algebraically.

The generalization to meromorphic finite-rank operators is trivial.

Proof of the theorem. Let $\bar{D}$ by any closed subdomain of D containing l_0. By means of the Heine-Borel theorem we have a finite set of domains $D_1, D_2, \cdots$ such that

$$\bigcup D_i=\bar{D}\,,$$
$$D_i\cap D_{i+1}\neq 0\,,$$

and $\|A(l')-A(l)\|<\frac{1}{2}$ provided l and l' belong to the same D_i.

Since $A(l)$ is completely continuous in D, we take a fixed point l_i in D_i and decompose

$$A(l_i)=S_i+N_i\,,$$

where S_i is of finite rank and $\|N_i\|<\frac{1}{2}$. For every l in D_i we then have

$$\|A(l)-S_i\|\leqslant\|A(l)-A(l_i)\|+\|N_i\|<1\,,$$

so that $[1-A(l)+S_i]^{-1}=[1-M(l)+P(l)+S_i]^{-1}$ exists and is analytic in D_i. We write the formal identity

$$[1-M(l)]^{-1} =\{1-[1-M(l)+P(l)+S_i]^{-1}[P(l)+S_i]\}^{-1} \times[1-M(l)+P(l)+S_i]^{-1}.$$

The operator $[1-M(l)+P(l)+S_i]^{-1}[P(l)+S_i]$ is of finite rank and meromorphic in D_i. Through the representation given by the lemma we can explicitly compute its resolvent in the form

$$1+\frac{1}{d(l)}\sum_{i=1}^{N}|\psi_i(l)\rangle\langle\varphi_i(l)|\,,$$

where $\psi_i(l)$ and $\varphi_i^*(l)$ are analytic in D and $d(l)$ is also an analytic function in D. The case $d(l)\equiv 0$ in D_i is excluded because it would imply that $[1-M(l)]^{-1}$ does not exist at any point of D_i and therefore of $\bar{D}$ because of the overlapping of the D_i's. Thus

$$[1-M(l)]^{-1}=[1-M(l)+P(l)+S_i]^{-1} +d^{-1}(l)\sum_{i=1}^{N}|\psi_i(l)\rangle\langle\varphi_i(l)| \times[1-M(l)+P(l)+S_i]^{-1}$$

which is the required form.

Note. Because of the identity

$$(1-M)^{-1}=(1+M+M^2+\cdots M^{n-1})(1-M^n)^{-1}$$

the theorem is also true if only some power M^n of our operator family satisfies the conditions.